Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 2, pp. 241–249, 2007

STRESS WAVES IN A ROD SUBJECTED TO A MOVING LOAD

R. G. Yakupov

UDC 539.3

The wave processes in a semi-infinite rod located in an elastic medium and subjected to a point load moving at a constant velocity are considered. The system of two differential equations of motion of Timoshenko beam theory is solved using the Laplace transform in time. The integrals obtained are determined numerically. Variation of the bending moment on the longitudinal coordinate behind the elastic-wave front and the region of action of the point force at various times is shown. The results of the solution are influence functions.

Key words: *rod, point load, stress waves.*

This paper analyzes longitudinal and transverse waves that arise in a semi-infinite rod located in an elastic medium and subjected to a point force moving at a constant velocity. Recently, the problem of self-propagation of stress and strain waves in engineering facilities of large extension has increased in importance.

The dynamics of engineering systems subjected to moving loads has been studied in many papers, which are reviewed in [1–3]. The conventional equations of motion of bending theory give approximate solutions of the wave problem for large times and correspond to simultaneous perturbation propagation along the rod [3]. In the case considered, the equation of motion is written with allowance for shear strains and the inertia of rotation [4]:

$$\frac{\partial Q}{\partial x} + \alpha W = p(x,t) - \rho F \frac{\partial^2 W}{\partial t^2}, \qquad \frac{\partial M}{\partial x} - Q = \rho I \frac{\partial^2 \theta}{\partial t^2}; \tag{1}$$

$$Q = k' GF\Big(\theta - \frac{\partial W}{\partial x}\Big), \qquad M = EJ_y \frac{\partial \theta}{\partial x}. \tag{2}$$

Here Q and M are the shear force and the bending moment, respectively, W is the deflection, $p(x,t)$ is the external force, ρ, E, and G are the density and elastic and shear moduli of the rod material, respectively, I is the polar moment of inertia of an element, F and J_y are the cross-sectional area and the axial moment of inertia, respectively, x is the longitudinal coordinate reckoned from the mounting device, t is time, k' is the cross-section shape coefficient ($k' = 1.2$ for a rectangular cross section and $k' = 1.1$ for a circular cross section), and α is the coefficient of the base defined by the formula [5]

$$\alpha = 0.12 E_* (b/l_0)^{1/2}/(1-\mu_*^2);$$

b is the cross-sectional width of the rod and l_0 is unit length. In the case of a rod of circular cross section $b = D$ (D is the rod diameter). The total angle of rotation is equal to

$$\frac{\partial W}{\partial x} = \theta + \beta_* \tag{3}$$

(θ and β_* are the angles of rotation due to the bending moment and shear force, respectively). Substitution of (2) and (3) into (1) yields the following equations in displacements:

Institute of Mechanics, Ufa Scientific Center, Russian Academy of Sciences, Ufa 450054; imran@anrb.ru.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 2, pp. 112–122, March–April, 2007. Original article submitted June 8, 2006.

$$k'GF\Big(\frac{\partial\theta}{\partial x}-\frac{\partial^2 W}{\partial x^2}\Big)+\rho F\,\frac{\partial^2 W}{\partial t^2}+\alpha W=p(x,t),\qquad EJ_y\,\frac{\partial^2\theta}{\partial x^2}-k'GF\Big(\theta-\frac{\partial W}{\partial x}\Big)-\rho I\,\frac{\partial^2\theta}{\partial t^2}=0. \tag{4}$$

Using the dimensionless quantities

$$\xi=\frac{x}{r},\quad \tau=\frac{c_1t}{r},\quad w=\frac{W}{r},\quad m=\frac{Mr}{EJ_y}$$

and introducing the notation

$$c_1^2=\frac{E}{\rho},\quad c_2^2=\frac{k'G}{\rho},\quad r^2=\frac{J_y}{F},\quad \gamma=\frac{c_1^2}{c_2^2},$$

we divide Eq. (4) by $k'GF$. As a result, we obtain

$$\begin{gathered}\frac{\partial^2 w}{\partial\xi^2}-\frac{\partial\theta}{\partial\xi}-\gamma\,\frac{\partial^2 w}{\partial\tau^2}-\zeta w=-\frac{rp(\xi,\tau)}{\rho Fc_2^2},\\ \frac{\partial w}{\partial\xi}-\theta+\gamma\Big(\frac{\partial^2\theta}{\partial\xi^2}-\frac{\partial^2\theta}{\partial\tau^2}\Big)=0.\end{gathered} \tag{5}$$

Here $\zeta=r^2\alpha/(\rho Fc_2^2)$ and c_1 and c_2 are the propagation velocities of the bending and shear waves. It is assumed that the external force is a point one and moves at a velocity V. The pressure function is written as

$$p(\xi,\tau)=p_0\delta(x-Vt)=p_0\delta[r(\xi-\tau/\beta)].$$

Here $\beta=c_1/V$ and δ is the Dirac delta function defined by the condition

$$\int_a^b f(\xi)\delta(\xi-X)\,d\xi=f(X),\qquad a\leqslant X\leqslant b.$$

For the rod at rest, where the force p_0 has not yet begin to move, the initial conditions are given by

$$\tau=0:\qquad w(\xi,0)=\theta(\xi,0)=0,\quad \frac{\partial w}{\partial\tau}=\frac{\partial\theta}{\partial\tau}=0.$$

We apply the Laplace transformation in time to system (5):

$$\frac{d^2\bar w}{d\xi^2}-(\gamma s^2+\zeta)\bar w-\frac{d\bar\theta}{d\xi}=-\frac{p_0r}{\rho Fc_2^2}\int_0^\infty \mathrm{e}^{-s\tau}\,\delta[r(\xi-\tau/\beta)]\,d\tau,\qquad \frac{d\bar w}{d\xi}+\gamma\,\frac{d^2\bar\theta}{d\xi^2}-(\gamma s^2+1)\bar\theta=0 \tag{6}$$

($\bar w$ and $\bar\theta$ are the images of the functions w and θ). On the right of the first formula in (6), we make the change of variable by the formula

$$z=r(\xi-r/\beta), \tag{7}$$

whence we obtain

$$\tau=\beta(\xi-z/r),\qquad d\tau=-(\beta/r)\,dz. \tag{8}$$

We substitute (7) and (8) into the integrand on the right of the first equation in (6) and perform integration:

$$\int_0^\infty \mathrm{e}^{-s\tau}\,\delta\Big[r\Big(\xi-\frac{\tau}{\beta}\Big)\Big]\,d\tau=-\frac{\beta}{r}\,\mathrm{e}^{-\beta s\tau}\,\mathrm{e}^{\beta sz}\Big|_{z=0}=-\frac{\beta}{r}\,\mathrm{e}^{-\beta s\xi}.$$

As a result, system (6) becomes

$$\begin{gathered}\frac{d^2\bar w}{d\xi^2}-(\gamma s^2+\zeta)\bar w-\frac{d\bar\theta}{d\xi}=k\gamma\,\mathrm{e}^{-\beta s\xi},\\ \frac{d\bar w}{d\xi}+\gamma\,\frac{d^2\bar\theta}{d\xi^2}-(\gamma s^2+1)\bar\theta=0,\end{gathered} \tag{9}$$

where $k=p_0/(\rho FVc_1)$.

Eliminating $\bar\theta$ from (9), we obtain

$$\frac{d^4\bar w}{d\xi^4}-[(\gamma+1)s^2+\zeta]\,\frac{d^2\bar w}{d\xi^2}+\Big[\gamma s^4+(\zeta+1)s^2+\frac{\zeta}{\gamma}\Big]\bar w=kf_2\,\mathrm{e}^{-\beta s\xi}\,. \tag{10}$$

Here $f_2=(\beta^2-1)\gamma s^2-1$. The solution of system (9) has the form

$$\begin{gathered}\bar w=A_1\,\mathrm{e}^{-\lambda_1\xi}+A_2\,\mathrm{e}^{-\lambda_2\xi}+D_1\,\mathrm{e}^{-\beta s\xi},\\ \bar\theta=(\lambda_1^2-\gamma s^2-\zeta)A_1\,\mathrm{e}^{-\lambda_1\xi}/(-\lambda_1)+(\lambda_2^2-\gamma s^2-\zeta)A_2\,\mathrm{e}^{-\lambda_2\xi}/(-\lambda_2)+D_2\,\mathrm{e}^{-\beta s\xi},\end{gathered} \tag{11}$$

where A_1 and A_2 are the constants of integration of Eq. (10), $D_1=kf_2/(b_1f_1)$, $D_2=k\beta s/(b_1f_1)$, $f_1=(s^2-a_3^2)(s^2-a_4^2)$, $a_{3,4}=[b_2\pm(b_2^2-4b_1\zeta/\gamma)^{1/2}]/(2b_1)$, $b_1=(\beta^2-\gamma)(\beta^2-1)$, $b_2=(\beta^2-1)\zeta-1$, and $\lambda_{1,2}$ are two (of the four) roots of the characteristic equation

$$\lambda^4-[(\gamma+1)s^2+\zeta]\lambda^2+[\gamma s^4+(\zeta+1)s^2+\zeta/\gamma]=0,$$

that satisfy the damping condition for $\bar w$ and $\bar\theta$ at infinity:

$$\lambda_{1,2}=(1/\sqrt2\,)\Big([(\gamma+1)s^2+\zeta]\pm\{(\gamma-1)^2s^4+2[(\gamma-1)\zeta-2]s^2+\zeta(\zeta-4/\gamma)\}^{1/2}\Big)^{1/2}. \tag{12}$$

Equation (12) is represented as

$$\lambda_{1,2}^2=[(\gamma+1)s^2+\zeta]/2\pm f/a, \tag{13}$$

where $f=[(s^2-a_1^2)(s^2-a_2^2)]^{1/2}$, $a_{1,2}^2=(a/2)\{(a-\zeta)\pm a[1-(\gamma-1)\zeta/\gamma]^{1/2}\}$, and $a=2/(\gamma-1)$. The function $\lambda_1^2(s)$ is analytic in the integration plane and is not equal to zero. Following [6], we multiply and divide λ_2^2 by λ_1^2. After transformations, we have

$$\lambda_2^2\lambda_1^2/\lambda_1^2=\gamma(s^2+\zeta/\gamma)(s^2+1/\gamma)/\lambda_1^2. \tag{14}$$

From (13) and (14), it follows that the branch points of λ_1 are the points $s=\pm a_1$, $s=\pm ia_2$, and the branch points of λ_2 are the points $s=\pm a_1$, $s=\pm ia_2$ and $s=\pm i(\zeta/\gamma)^{1/2}$, $s=\pm i(1/\gamma)^{1/2}$.

Let us consider two types of mounting device in the initial cross section: hinged fastening and clamping. For hinged fastening, the transformed boundary conditions are written as

$$\xi=0:\qquad \bar w(0,\tau)=\frac{\partial\bar\theta}{\partial\xi}=0.$$

The integration constants are equal to

$$A_1=\frac{ka}{2f}\Big[\gamma+(\lambda_2^2-\beta^2s^2)\,\frac{f_2}{b_1f_1}\Big],\qquad A_2=-\frac{ka}{2f}\Big[\gamma+(\lambda_1^2-\beta^2s^2)\,\frac{f_2}{b_1f_1}\Big].$$

In the case of clamping, we have zero rotation and shear angles

$$\xi=0:\qquad \bar\theta=0,\quad \frac{\partial\bar w}{\partial\xi}-\bar\theta=0,$$

and the integration constants are equal to

$$A_1=\frac{ak\lambda_1\lambda_2^2}{2\beta fs}\Big[\frac{\gamma}{\gamma s^2+\beta}+\frac{f_2}{b_1f_1}\Big],\qquad A_2=-\frac{ak\lambda_2\lambda_1^2}{2\beta fs}\Big[\frac{\gamma}{\gamma s^2+\beta}+\frac{f_2}{b_1f_1}\Big].$$

The image of the bending moment is defined by the formula

$$\bar m=A_1^*\,\mathrm{e}^{-\lambda_1\xi}+A_2^*\,\mathrm{e}^{-\lambda_2\xi}-D_3^*\,\mathrm{e}^{-\beta s\xi}, \tag{15}$$

where $A_1^*=(\lambda_1^2-\gamma s^2-\zeta)A_1$, $A_2^*=(\lambda_2^2-\gamma s^2-\zeta)A_2$, and $D_3^*=k\beta^2s^2/(b_1f_1)$.

The inversion formula is written as

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}F(s)\,\mathrm{e}^{\tau s}\,ds=\begin{cases}f(\tau), & \tau>0,\\ 0, & \tau<0.\end{cases}$$

Let us determine the original of the bending moment in the case of hinged fastening for $V<c_2$. According to (15), we write

$$m(\xi,\tau) = (I_1 + I_2 + I_3)/(2\pi i). \tag{16}$$

Here

$$I_1 = \int\limits_{c-i\infty}^{c+i\infty} A_2^* \,\mathrm{e}^{(\tau-\lambda_2\xi/s)s}\, ds, \qquad I_2 = \int\limits_{c-i\infty}^{c+i\infty} A_2^* \,\mathrm{e}^{(\tau-\lambda_1\xi/s)s}\, ds, \qquad I_3 = \int\limits_{c-i\infty}^{c+i\infty} D_3^* \,\mathrm{e}^{(\tau-\beta\xi)s}\, ds.$$

The integrands in (16) have the following property:

$$s \to \infty: \qquad A_1^*(s) \to 0, \quad A_2^*(s) \to 0, \quad D_3^*(s) \to 0.$$

In this case, λ_1 and λ_2 tend to constant values:

$$\lim_{s\to\infty} \frac{\lambda_2}{s} = 1, \qquad \lim_{s\to\infty} \frac{\lambda_1}{s} = \sqrt{\gamma}.$$

At any specified time τ, the integrals I_1 and I_2 are not equal to zero for $\xi < \tau$ and $\xi < \tau/\sqrt{\gamma}$, respectively, and are equal to zero for $\xi > \tau$ and $\xi > \tau/\sqrt{\gamma}$, respectively [6]. Similarly, $I_3 \neq 0$ in the case $\xi < \tau/\beta$ and $I_3 = 0$ in the case $\xi > \tau/\beta$. The region of perturbation propagation is divided by the bending- and shear-wave fronts and the point force p_0 into three parts. The coordinates of the wave fronts are $\xi_1 = \tau$ and $\xi_2 = \tau/\sqrt{\gamma}$, and the coordinate of the point force is $\xi_3 = \tau/\beta$. The entire region $0 < \xi < \xi_1$ is encompassed by the bending wave. The wave parameters are determined using the integral I_1. In the interval $0 < \xi < \xi_2$, bending and shear waves are present, and the shear-wave parameters are defined by the integral I_2. In the region $0 < \xi < \xi_3$, in addition to bending and shear strains, there are strains due to the action of the point force, which are defined by the integral I_3. In the interval $0 < \xi < \xi_3$, all types of strain are present.

For convenience in converting to real integrals, the expression for I_1 and I_2 are written as the sum

$$\frac{1}{2\pi i}\, I_1 = I_1' + I_1'', \qquad \frac{1}{2\pi i}\, I_2 = I_2' + I_2'',$$

where

$$I_1' = -\frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{ka\gamma}{2f}\, (\lambda_2^2 - \gamma s^2 - \zeta)\, \mathrm{e}^{\tau s - \lambda_2 \xi}\, ds, \qquad \tau > \xi,$$

$$I_1'' = -\frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{ka}{2b_1}\, \frac{[(\beta^2-1)\gamma s^2 - 1](\lambda_1^2 - \beta^2 s^2)(\lambda_2^2 - \gamma s^2 - \zeta)\, \mathrm{e}^{\tau s - \lambda_2\xi}}{f f_1}\, ds, \qquad \xi < \tau < \sqrt{\gamma}\,\xi,$$

$$I_2' = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{ka\gamma}{2f}\, (\lambda_1^2 - \gamma s^2 - \zeta)\, \mathrm{e}^{\tau s - \lambda_1 \xi}\, ds, \qquad \sqrt{\gamma}\,\xi < \tau < \beta\xi,$$

$$I_2'' = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{ka}{2b_1}\, \frac{[(\beta^2-1)\gamma s^2 - 1](\lambda_2^2 - \beta^2 s^2)(\lambda_1^2 - \gamma s^2 - \zeta)\, \mathrm{e}^{\tau s - \lambda_1\xi}}{f f_1}\, ds, \qquad \sqrt{\gamma}\,\xi < \tau < \beta\xi,$$

$$\frac{1}{2\pi i}\, I_3 = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{k\beta^2 s^2}{b_1 f_1}\, \mathrm{e}^{(\tau-\beta\xi)s}\, ds, \qquad \tau > \beta\xi.$$

In the integral I_1, the integrands have the branch points $s = \pm a_1$, $\pm i a_2$, $\pm i a_5$, and $\pm i a_6$; in the integral I_2, the branch points are $s = \pm a_1$ and $\pm i a_2$. In addition to the indicated branch points, the functions in I_1'' and I_2'' have simple poles at the points $\pm i a_3$ and $\pm i a_4$.

The contour integrals are converted to real integrals. The contours for the integration of I_1 and I_2 are presented in Fig. 1. The complex expressions in the integrands are calculated subject to the constraints on their arguments depending on the integration path and are given in Table 1. The calculations were made using the formula

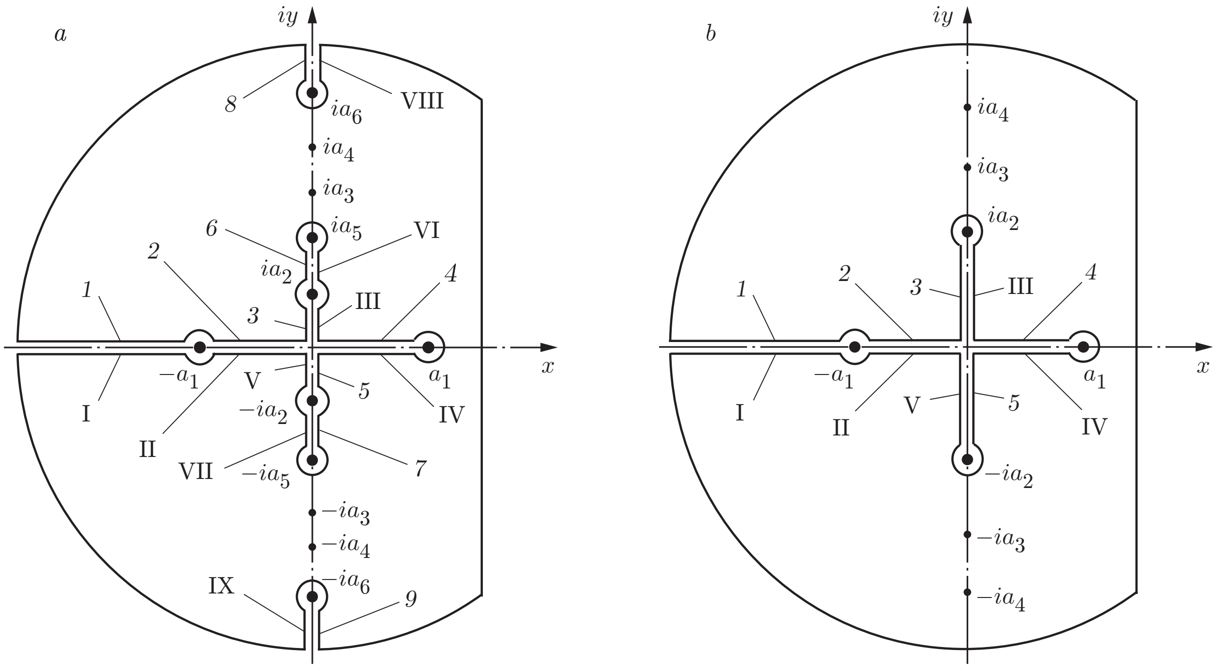

Fig. 1. Contours of integration of I_1 (a) and I_2 (b) by formula (16): 1–9 and I–IX are the opposite faces of the integration path.

$$I = \sum \operatorname{res}(s) - \sum \int\limits_{\gamma_i},$$

where γ_i are the integration paths on the cut faces and a circular arc of infinitesimal radius. As the circle radius tends to zero, the integrals vanish. The calculations yield

$$I_1' = \frac{ka\gamma}{2\pi} \int\limits_0^{a_1} \Big(\frac{\mathrm{e}^{-(\tau x+\eta_1\xi)}[(R_1 - \gamma x^2 - \zeta)\cos(\eta_2\xi) + R_2 \sin(\eta_2\xi)]}{\sqrt{(a_1^2 - x^2)(a_2^2 + x^2)}}$$

$$+ \frac{\mathrm{e}^{\tau x-\eta_2\xi}[(R_1 - \gamma x^2 - \zeta)\cos(\eta_1\xi) + R_2 \sin(\eta_1\xi)]}{\sqrt{(a_1^2 - x^2)(a_2^2 + x^2)}} \Big)\, dx,$$

$$I_1'' = \operatorname{res}(s)_1 - \frac{ka}{2\pi b_1} \int\limits_0^{a_1} \Big(\frac{\mathrm{e}^{-(\tau x+\eta_1\xi)}[(\beta^2 - 1)\gamma x^2 - 1][T_1 \cos(\eta_2\xi) - T_2 \sin(\eta_2\xi)]}{\sqrt{(a_1^2 - x^2)(a_2^2 + x^2)}\,(a_3^2 + x^2)(a_4^2 + x^2)}$$

$$+ \frac{\mathrm{e}^{\tau x-\eta_2\xi)}[(\beta^2 - 1)\gamma x^2 - 1][T_1 \cos(\eta_1\xi) - T_2 \sin(\eta_1\xi)]}{\sqrt{(a_1^2 - x^2)(a_2^2 + x^2)}\,(a_3^2 + x^2)(a_4^2 + x^2)} \Big)\, dx,$$

$$\operatorname{res}(s)_1 = \frac{ka}{2b_1} \Big(\frac{[(\beta^2 - 1)\gamma a_3^2 + 1](\bar{R}_1 - \bar{R}_2 + \beta^2 a_3^2)(\bar{R}_1 + \bar{R}_2 + \gamma a_3^2 - \zeta)\,\mathrm{e}^{-\eta_6\xi} \sin(a_3\tau)}{a_3(a_3^2 - a_4^2)\sqrt{(a_3^2 + a_1^2)(a_3^2 - a_2^2)}}$$

$$+ \frac{[(\beta^2 - 1)\gamma a_4^2 + 1](\bar{R}_1 - \bar{R}_2 + \beta^2 a_4^2)(\bar{R}_1 + \bar{R}_2 + \gamma a_4^2 - \zeta)\,\mathrm{e}^{-\eta_6\xi} \sin(a_4\tau)}{a_4(a_4^2 - a_3^2)\sqrt{(a_4^2 + a_1^2)(a_4^2 - a_2^2)}} \Big),$$

$$R_1 = [(\gamma + 1)x^2 + \zeta]/2, \quad R_2 = (\gamma + 1)[|a_2^2 - x^2|(a_2^2 + x^2)]^{1/2}/2, \quad R = (R_1^2 + R_2^2)^{1/2},$$

$$T_1 = (R_1 - \beta^2 x^2)(R_1 - \gamma x^2 - \zeta) + R_2^2, \quad T_2 = R_2[(\beta^2 - \gamma)x^2 - \zeta],$$

TABLE 1

Complex Quantities in the Expressions of the Integrals I_1 and I_2

Integration path	s	s^2	$s+a_1$	$s-a_1$	$\sqrt{s+a_1}\,\sqrt{s-a_1}$	$s+ia_2$
$\frac{1}{\mathrm{I}}$	$\lvert x\rvert\,\mathrm{e}^{\pm i\pi}$	x^2	$(x+a_1)\,\mathrm{e}^{\pm i\pi}$	$(x-a_1)\,\mathrm{e}^{\pm i\pi}$	$-\sqrt{x^2-a_1^2}$	$\sqrt{a_2^2+x^2}\;\mathrm{e}^{i(\pi-\alpha)}$
$\frac{2}{\mathrm{II}}$	$\lvert x\rvert\,\mathrm{e}^{\pm i\pi}$	x^2	$(a_1-x)\,\mathrm{e}^{\pm i\pi}$	$(a_1-x)\,\mathrm{e}^{\pm i\pi}$	$\pm i\sqrt{a_1^2-x^2}$	$\sqrt{a_2^2+x^2}\;\mathrm{e}^{i(\pi-\alpha)}$
$\frac{4}{\mathrm{IV}}$	x	x^2	$(a_1-x)\,\mathrm{e}^{\pm i\pi}$	$(a_1-x)\,\mathrm{e}^{\pm i\pi}$	$\pm i\sqrt{a_1^2-x^2}$	$\sqrt{a_2^2+x^2}\;\mathrm{e}^{i\alpha}$
$\frac{3}{\mathrm{III}}$	iy	$-y^2$	$\sqrt{a_1^2+y^2}\,\frac{\mathrm{e}^{i\alpha}}{\mathrm{e}^{i(\pi-\alpha)}}$	$\sqrt{a_1^2+y^2}\,\frac{\mathrm{e}^{i(\pi-\alpha)}}{\mathrm{e}^{i\alpha}}$	$i\sqrt{a_1^2+y^2}$	$(a_2+y)\,\mathrm{e}^{i\pi/2}$
$\frac{5}{\mathrm{V}}$	$-iy$	$-y^2$	$\sqrt{a_1^2+y^2}\;\mathrm{e}^{-i\alpha}$	$\sqrt{a_1^2+y^2}\;\mathrm{e}^{-i(\pi-\alpha)}$	$-i\sqrt{a_1^2+y^2}$	$(a_2+y)\,\mathrm{e}^{-i\pi/2}$
$\frac{6,\ \mathrm{VI}}{7,\ \mathrm{VII}}$	$\pm iy$	$-y^2$	$\sqrt{a_1^2+y^2}\,\frac{\mathrm{e}^{i\alpha}}{\mathrm{e}^{-i\alpha}}$	$\sqrt{a_1^2+y^2}\;\mathrm{e}^{\pm i(\pi-\alpha)}$	$\pm i\sqrt{a_1^2+y^2}$	$(y\pm a_2)\,\mathrm{e}^{\pm i\pi/2}$

Note. Arabic and Roman numerals correspond to segments of the integration contour in Fig. 1.

$$\bar{R}_1=[\zeta-(\gamma+1)y^2]/2,\quad \bar{R}_2=(\gamma-1)[(a_1^2+y^2)|a_2^2-y^2|]^{1/2}/2,\quad \bar{R}=(\bar{R}_1^2+\bar{R}_2^2)^{1/2},$$

$$\eta_{1,2}=[(R\mp R_1)/2]^{1/2},\quad \eta_{3,4}=[(\bar{R}\pm\bar{R}_1)/2]^{1/2},\quad \eta_{5,6}=|\bar{R}_1\mp\bar{R}_2|^{1/2}.$$

In the first term of the expression for $\mathrm{res}\,(s)_1$, the quantities $\bar{R}_1$, $\bar{R}_2$, and η_6 are determined for $y=a_3$, and in the second term, they are determined for $y=a_4$.

For I_2, the following relations hold:

$$I_2'=-I_1',\qquad I_2''=-(I_2'-\mathrm{res}\,(s)_1)+\mathrm{res}\,(s)_2,$$

$$\mathrm{res}\,(s)_2=\frac{ka}{2b_1}\Big(\frac{T_3[(\beta^2-1)\gamma a_3^2+1]\sin(a_3\tau)\cos(\eta_5\xi)}{a_3(a_4^2-a_3^2)\sqrt{(a_3^2+a_1^2)(a_3^2-a_2^2)}}$$

$$+\frac{T_3[(\beta^2-1)\gamma a_4^2+1]\sin(a_4\tau)\cos(\eta_5\xi)}{a_4(a_3^2-a_4^2)\sqrt{(a_4^2+a_1^2)(a_4^2-a_2^2)}}\Big),$$

$$T_3=(\bar{R}_1+\bar{R}_2+\beta^2y^2)(\bar{R}_1-\bar{R}_2+\gamma y^2-\zeta).$$

In first term of the expression for $\mathrm{res}\,(s)_2$, the quantities $\bar{R}_1$, $\bar{R}_2$, η_5, and T_3 are determined for $y=a_3$, in the second term, they are determined for $y=a_4$.

The integral I_3 has simple poles at the points $\pm ia_3$, $\pm ia_4$ and is equal to the sum of residues at the poles:

$$\frac{1}{2\pi i}\,I_3=\mathrm{res}\,(s)_3,\qquad \tau>\beta\xi.$$

Here

$$\mathrm{res}\,(s)_3=\frac{k}{b_1(a_4^2-a_3^2)}\,[\beta^2a_4\sin((\tau-\beta\xi)a_4)-\beta^2a_3\sin((\tau-\beta\xi)a_3)].$$

$s - ia_2$	$\sqrt{s+ia_2}\times\sqrt{s-ia_2}$	f	$\lambda^2_{1,2}$	$\lambda_{1,2}$	Interval of variation x, y
$\sqrt{a_2^2+x^2}\ \mathrm{e}^{-i(\pi-\alpha)}$	$\sqrt{a_2^2+x^2}$	$-\sqrt{x^2-a_1^2}\times\sqrt{x^2+a_2^2}$	$R_1 \mp R_2$	$\sqrt{R_1 \mp R_2}$	$x < -a_1$
$\sqrt{a_2^2+x^2}\,\mathrm{e}^{-i(\pi-\alpha)}$	$\sqrt{a_2^2+x^2}$	$\pm i\sqrt{a_1^2-x^2}\times\sqrt{a_2^2+x^2}$	$\dfrac{R_1 \pm R_2}{R_1 \mp R_2}$	$\dfrac{\eta_1 \pm i\eta_2}{\eta_1 \mp i\eta_2}$	$-a_1 < x < 0$
$\sqrt{a_2^2+x^2}\ \mathrm{e}^{-i\alpha}$	$\sqrt{a_2^2+x^2}$	$\pm i\sqrt{a_1^2-x^2}\times\sqrt{a_2^2+x^2}$	$\dfrac{R_1 \pm R_2}{R_1 \mp R_2}$	$\dfrac{\eta_2 \pm i\eta_1}{\eta_2 \mp i\eta_1}$	$0 \leqslant x \leqslant a_1$
$(a_2-y)\,\mathrm{e}^{-i\pi/2}$	$\sqrt{a_2^2-y^2}$	$i\sqrt{a_1^2+y^2}\times\sqrt{a_2^2-y^2}$	$\bar{R}_1 \pm i\bar{R}_2$	$\eta_3 \pm i\eta_4$	$0 \leqslant y \leqslant a_2$
$(a_2-y)\,\mathrm{e}^{i\pi/2}$	$\sqrt{a_2^2-y^2}$	$-i\sqrt{a_1^2+y^2}\times\sqrt{a_2^2-y^2}$	$\bar{R}_1 \mp i\bar{R}_2$	$\eta_3 \mp i\eta_4$	$-a_2 \leqslant y \leqslant 0$
$(y \mp a_2)\,\mathrm{e}^{\pm i\pi/2}$	$\pm i\sqrt{y^2-a_2^2}$	$-\sqrt{y^2+a_1^2}\times\sqrt{y^2-a_2^2}$	$\bar{R}_1 \mp \bar{R}_2$	$\lambda_1 = i\eta_5$ $\lambda_2 = \eta_6$	$y < -a_2$

The integrals over the segments of the integration contour denoted by figures 1 and I and over the entire cut faces along the imaginary axis are mutually cancelled.

For the particular case $V = c_2$, in (11) it is necessary to set

$$D_1 = k[(\gamma-1)\gamma s^2 - 1]/[b_3(s^2+a_7^2)], \qquad D_2 = k\sqrt{\gamma}\, s/[b_3(s^2+a_7^2)],$$

where $b_3 = 1-(\gamma-1)\xi$ and $a_7^2 = \zeta/(b_3\gamma)$. In the case of hinged fastening, the bending moment is expressed as

$$m(\xi,\tau) = (I_1+I_2+I_3)/(2\pi i), \qquad \tau > \sqrt{\gamma}\xi.$$

Here

$$I_1 = -\frac{ka}{2}\int_{c-i\infty}^{c+i\infty}\Big[\frac{\gamma}{f} + (\lambda_1^2-\gamma s^2)\frac{F_1}{F_2}\Big]F_3\,\mathrm{e}^{\tau s-\lambda_2\xi}\,ds, \qquad I_2 = \frac{ka}{2}\int_{c-i\infty}^{c+i\infty}\Big[\frac{\gamma F_5}{f} + (\lambda_2^2-\gamma s^2)\,\frac{F_1F_4}{F_2}\Big]\,\mathrm{e}^{\tau s-\lambda_1\xi}\,ds,$$

$$F_1 = (\gamma-1)\gamma s^2 - 1, \quad F_2 = fb_3(s^2+a_7^2), \quad F_3 = \lambda_2^2-\gamma s^2-\zeta, \quad F_4 = \lambda_1^2-\gamma s^2-\zeta, \quad F_5 = \lambda_1^2-\gamma s^2-1,$$

$$I_3 = k\int_{c-i\infty}^{c+i\infty}\frac{\gamma s^2}{b_3(s^2+a_7^2)}\,\mathrm{e}^{(\tau-\sqrt{\gamma}\xi)s}\,ds.$$

Let us consider a numerical example. We assume that $p_0 = 10$ kN/m, the cross section of the beam is rectangular, $b = h = 0.1$ m, $F = b\times h$, $E = 2\cdot 10^5$ MPa, $\rho = 8$ tons/m^3, $c_1 = 5\cdot 10^3$ m/sec, $c_2 = 2.84\cdot 10^3$ m/sec, $V = 2\cdot 10^3$ m/sec, $\zeta = 1.35\cdot 10^{-2}$, $\gamma = 3.1$, $\beta = 2.5$, $k = 1.25\cdot 10^{-6}$, $a = 0.95$, $a_1 = 0.94$, $a_2 = 0.051$, $a_3 = 0.072$, $a_4 = 0.225$, $a_5 = 0.066$, $a_6 = 0.568$, $a_7 = 0.067$, $b_1 = 16.6$, $b_2 = -0.929$, and $b_3 = 0.972$. The calculations were made using the formulas

$$m(\xi,\tau) = \begin{cases} I_1' + I_1'', & \xi_2 \leqslant \xi \leqslant \xi_1, \\ \mathrm{res}\,(s)_1 + \mathrm{res}\,(s)_2, & \xi_3 \leqslant \xi \leqslant \xi_2, \\ \mathrm{res}\,(s)_1 + \mathrm{res}\,(s)_2 + \mathrm{res}\,(s)_3, & 0 \leqslant \xi \leqslant \xi_3. \end{cases}$$

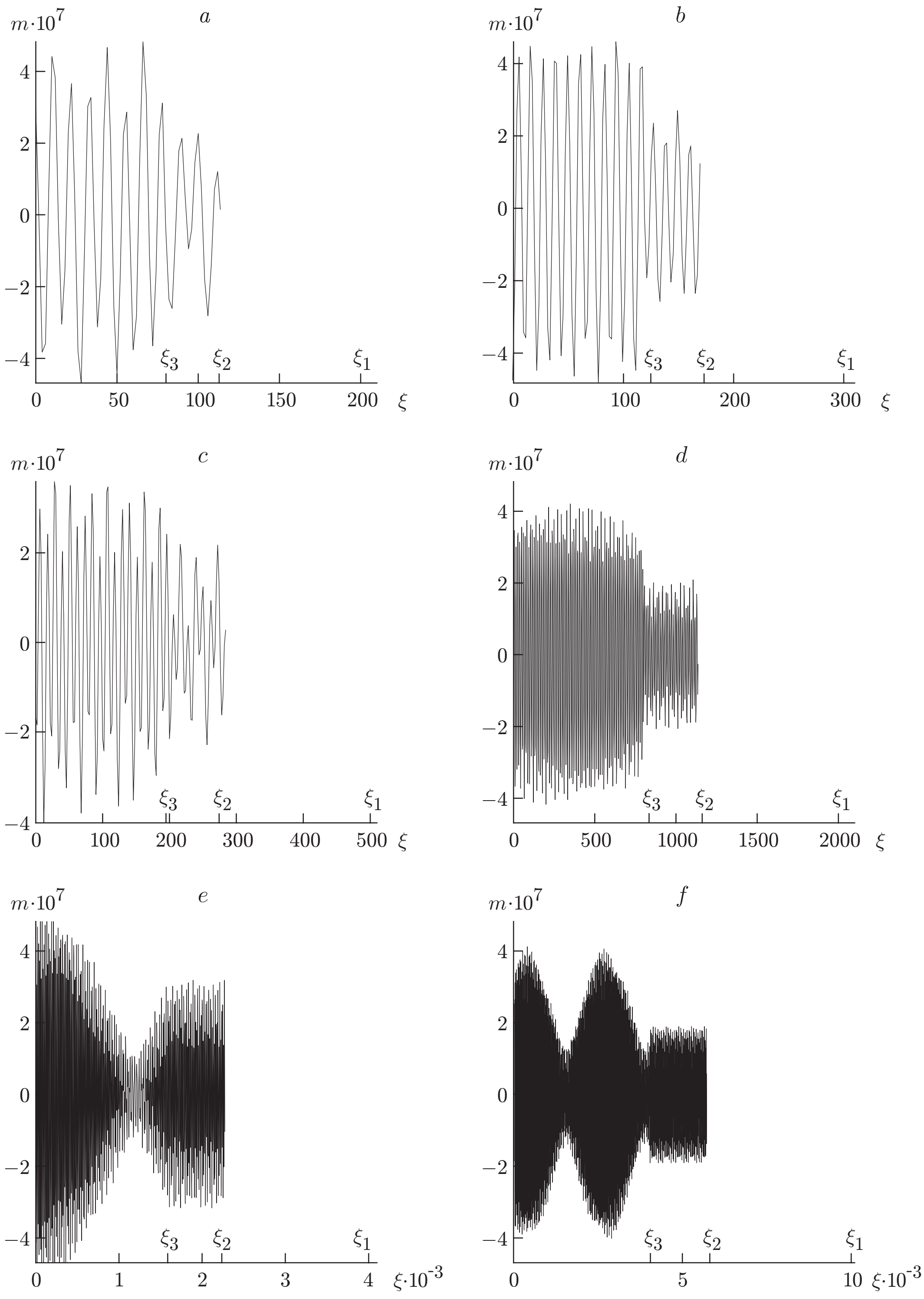

Fig. 2. Distribution of the bending moment along the rod for $V < c_2$ and $\tau = 200$ (a), 300 (b), 500 (c), 2000 (d), 4000 (e), and 10,000 (f).

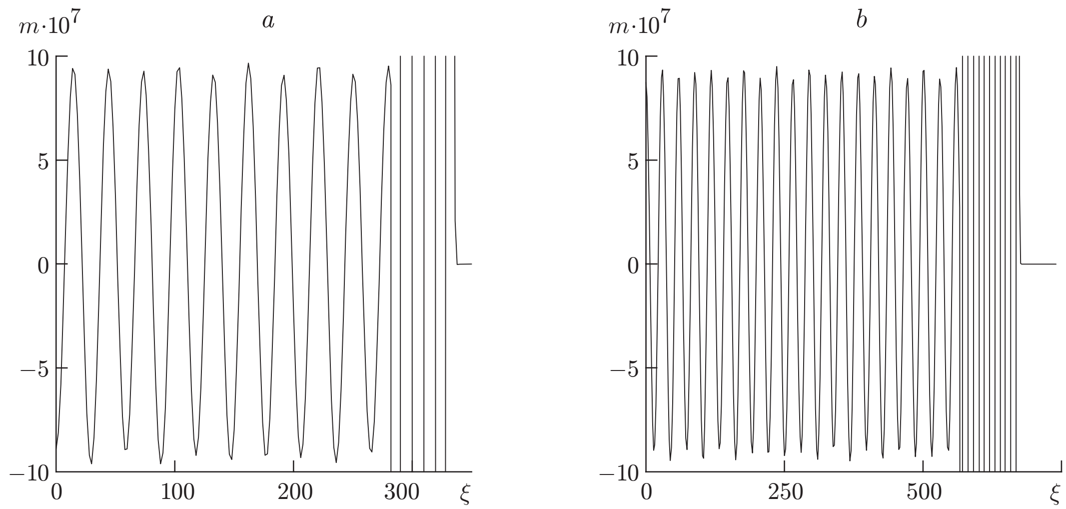

Fig. 3. Distribution of the bending moment along the rod for $V = c_2$ and $\tau = 500$ (a) and 1000 (b).

The integrands in the integrals I_1' and I_1'' are Oscillating, and at the point $x = a_1$, they have an infinite discontinuity; therefore, the integration over x was performed using not less than ten steps within the half-wavelength and setting the upper limit in the form $a_1(1 - \delta)$, where $\delta = 10^{-15}$. Thus, the principal values of the improper integrals were determined. The calculations were performed using the trapezoid method.

The calculation results were used to plot curves of $m(\xi)$ at various times (Figs. 2 and 3). For the observer located at a fixed point of the rod, the curves are oscillograms of the bending moment.

From the above data, it follows that in the perturbation zone, the motion of the rod has a complex vibrational nature. For $V < c_2$, the vibration frequency is $f = 9$ kHz, and in the particular case $V = c_2$, we have $f = 3.4$ kHz. The bending wave amplitude is two orders of magnitude smaller than the shear wave amplitude and is indiscernible in the given scale. The strains at the shear wave and behind the load are equal to 0.86 and 1.52 MPa, respectively.

Ahead of the shear wave front, there is a sharp increase in the amplitude of the bending moment m, which moves at a velocity c_2. The jump in the amplitude is due to the assumption of a point load [2]. As the load moves along the rod, the magnitude of the amplitude jump, the length of the rod segment on which there is the sudden increase in the amplitude, and, hence, the number of vibrations increase. As a result, each particle of the rod material is subjected to the bending moment which increases in time.

This work was supported by the Russian Foundation for Basic Research (Grant No. 05-01-97905).

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